LATERAL-TORSIONAL BUCKLING IN TAPERED SECTIONS

SITHINUN KITTIPRAPORN

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School of Engineering and Design, Brunel University

ABSTRACT

Tapered steel beams are adopted to optimise structural components in terms of weight, material usage and load capacity at each cross-section taking into account the respective distribution of stresses. The design guidance for non-uniform member such as tapered beams regarding the stability of the section is not provided in the Eurocode 3.

In this dissertation, the case of web-tapered beams is researched and tapered beams with taper ratio of $\gamma_h = 2$ were analysed. A linear analysis was performed to obtain the critical moment for lateral-torsional buckling. In results, a new reduction curve was plotted and compared with the reduction curve given in the Eurocode 3. Thus, a new equation for reduction factor is proposed.

ACKNOWLEDGEMENTS

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Lastly, I would like to say that I am grateful to have Dr. Cashell and Dr. Triantafyllou as my supervisor.

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LIST OF NOTATION

Lowercases:

a_0, a, b, c, d	Buckling curve
b	Cross-section width
b_{max}	Maximum cross-section width
b_{min}	Minimum cross-section width
e_0	Maximum amplitude of a member imperfection
f_y	Yield stress
h	Cross-section height
h_{max} or h_l	Maximum cross-section height
h_{min} or h_s	Minimum cross-section height
t_f	Thickness of flange
t_w	Thickness of web
v_0	Maximum initial imperfection
x-x	Axis along the member
y-y	Cross-section axis parallel to flanges
z-z	Cross-section axis perpendicular to flanges

Uppercases:

A	Cross-section area
AISC	American Institute of Steel Construction
A_{eff}	Effective cross sectional area
C_1	Correction factor
E	Modulus of elasticity
EC3	Eurocode 3
EC3-1-1	Eurocode 3 Part 1-1
FEM	Finite Element Method
G	Shear modulus
GMNIA	Geometrical and Material Non-linear Analysis with Imperfections
I	Second moment of area
I_T	Torsional constant
I_w	Warping constant
I_y, I_z	Second moment of area about the major and minor axis
$I_{y,eq}$	Equivalent 2nd moment of area, y-y axis
$I_{y,max}$	Maxmium 2nd moment of area, y-y axis
$I_{y,min}$	Minimum 2nd moment of are, y-y axis
L	Member length

L_{cr}	Length of the beam between points of lateral restraint
LTB	Lateral-torsional buckling
M	Bending moment
$M_{b,Rd}$	Design buckling resistance moment
$M_{y,Rd}$	Design bending moment resistance
N	Normal force
NCCI	Non-Contradictory Complementary Information
$N_{b,Rd}$	Design buckling resistance of compression member
N_{ed}	Design normal force
Q	Shear force
SCI	Steel Construction Institute
T	Torsion
UDL	Uniformly distributed loading
W_y	Section modulus, related to moment of inertia I, where W= I/y
$W_{el,y}$	Elastic section modulus
$W_{pl,y}$	Plastic section modulus

Lowercase Greek letters

α	Angle of taper
$\alpha_{cr,op}$	Minimum amplifier for the in-plane design loads to reach the elastic critical resistance with regard to lateral-torsional buckling
$\alpha_{ult,k}$	Minimum load amplifier of the design loads to reach the characteristic resistance of the most critical cross-section
β	Correction factor for the lateral-torsional buckling curves
γ_{M0}	Partial factor for resistance of cross-sections for all class
γ_{M1}	Partial safety factor for resistance of members to instability assessed by member checks
ε	Coefficient depending on f_y
λ	Eigenvalue
$\bar{\lambda}$	Non-dimensional slenderness
$\bar{\lambda}_{LT}$	Non-dimensional slenderness for lateral-torsional buckling
χ	Reduction factor
χ_{LT}	Reduction factor to lateral-torsional buckling
ψ	Stress ratio

Chapter 1

1.0: INTRODUCTION

1.1: Importance of the project

Structural members such as beams are essential and frequently used in. Eurocode 3 [1] provides structural design rules that may be applied to build steel structures and structural components, as well as other products of steel. For the common forms of construction, rules are provided and it is recommended that specialist advice is needed when considering an unusual structure. Moreover, EC3 serves as reference documents, which is recognized by the EU member states as a framework to develop harmonized technical specifications for construction products, as a basis for specifying contracts and as a means to prove compliance with the essential requirements of Council Directive 89/106/EEC.

Among its other applications, EC3 can be employed to verify the stability of various uniform structural members and frames. The stability of the uniform members and frames are in EC3 where the stability of uniform columns is check by the application of Clauses 6.3.1, and Clause 6.3.2 for uniform beams and Clause 6.3.3 for uniform beam-columns. However, *the design guidance for non-uniform member such as tapered section regarding the stability of the section is not provided in the EC3*. Because of this, the evaluation and verification of the buckling resistance for non-uniform members should be performed according to Clause 6.3.4 (General method for lateral and lateral torsional buckling of structural components) and GMNIA or numerical analysis, which accounts for geometrical nonlinearities. Regardless of the methods given, there are limitations in the process of verifying the strength capacity of a non-uniform member. These limitations are listed below,

i) Determining the cross section class

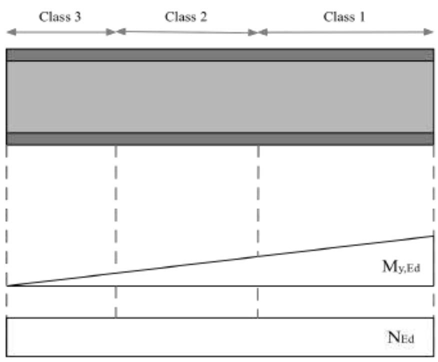


Figure 1-0: Uniform beam with non-uniform loading

EC3 defines the 4 classes of cross section. The highest class (cross sectional resistances) is chosen for a tapered member. The shape and magnitude of a beam's imperfections of the beam causes difficulties for the verification of a non-uniform member. Figure 1-0 illustrates a beam with different ratios of $M_{y,Ed}$ to N_{Ed} over the member length, thus, the cross section classification changes (Class 1,2 and 3) due to different loading. In addition, Figure 1-0 shows that $M_{y,Ed}$ for class 3 cross-section are not critical compared to $M_{y,Ed}$ in the rest of the beam. In the process of designing the beam, stresses must be evaluated along the beam to develop a design cross-section class. The highest cross-section class given will be used, thus, this design can result in an inefficient utilization of materials.

Identifying the critical cross-section (critical design location) can be a lengthy procedure. Despite the given equation for calculating the equivalent cross-section property of elastic critical forces for tapered members (length – Galambos 1998 [2] and depth – Galéa 1986 [3]), applying these equations in the buckling design equation is not validated for the tapered section.

Furthermore, when using the General Method, Finite Element Analysis may be used in determining $\alpha_{cr,op}$ and $\alpha_{ult,k}$ in the verification of the resistance to lateral torsional buckling. However, the method will have simplifications and may not cover the real behavior of the members.

ii) Choosing an adequate buckling curve

The choice to which buckling curve to use depends upon the geometry of the cross section. The appropriate buckling curve (imperfection factor α) is determined from Table 6.2 of Eurocode 3. For a tapered member varying cross-section, there can be various buckling curves. Buckling curve d or c as shown in Table 1-0 is directed to be used for non-uniform member in EC3, is adopted in the design, however, this may over predict the resistance level, as the buckling curve is different from that for the uniform members, and result in an over-conservative design. This means that the beam may be designed to resist much greater moment, than needed. In this design; the production of the beam would consume more materials than required which lead to a unsustainable use of materials. Furthermore, a specific buckling curve is applied to individual buckling cases in EC3 [1]. When the General Method is applied, choosing a buckling curve for tapered member can be a problem as height and width (*h/b)* varies along the section.

Table 1-0 present possible buckling curve for web-tapered beams:

Clause	Hot rolled	Welded
6.3.1	h/b ≤ 1.2 = Curve c	Curve c
6.3.2.2 (general case LTB)	h/b ≤ 2 = Curve a	h/b ≤ 2 = Curve c
	h/b > 2 = Curve b	h/b > 2 = Curve d
		where $\bar{\lambda}_{LT}$, =0.4 and β = 0.75
6.3.2.3 (special case LTB)	h/b ≤ 2 = Curve b	h/b ≤ 2 = Curve c
	h/b > 2 = Curve c	h/b > 2 = Curve d
		where, =0.4 and β = 0.75

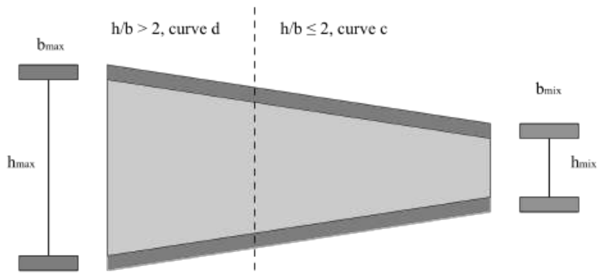


Figure 1-1: Tapered beam

Figure 1-1 shows, how the ratio *h/b* will be different across the section length, when considering a tapered section. The buckling curve with the highest imperfection factor α is chosen. This, therefore, leads to an inefficient design.

1.2: Objectives

In respect to the importance of this project, tapered steel members are advantageous and commonly used over prismatic members. The crucial advantage of the tapered steel members is the structural material's economization due to its utilization of structural material as optimizing the cross section can save material. With the limitation given above, the EC3 safety verification for tapered beam may lead to an over prediction of the material required for the member.

In this dissertation, tapered beam with varying linear web is studied and a proposed formula for lateral torsional buckling verification of web-tapered beam is given. The main objectives of the project are as follows:

- Applying and reviewing the existing method in EC3 for the buckling of uniform and non-uniform members
- Validating of General Method (clause 6.3.4) for stability checking of tapered member.
- Creating finite element models for tapered member and developing a reduction curve

1.3: The use of tapered members

Tapered members are commonly used in the steel construction industry, including continuous frame construction, typical products of which are single-level warehouses, exhibition centres and such. In addition to their structural efficiency, tapered members can be used to meet architectural requirements, making the structure to become more attractive.

Figures 1-3 to 1-5 illustrate the use of tapered section in bridges, while Figure 1-6 to 1-8 show the application of tapered section in building. Lastly, Figure 1-9 shows an example of the use of a tapered column.

Application in bridges:

Figure 1-3: Three Bridges over Hoofdvaart, Amsterdam, 2004; Tapered beam along the bridge length

Figure 1-4: La Devesa Bridge, Ripoll, Spain, 1991; Tapered arms that serve to transmit load from deck to arch

Figure 1-5: The Alamillo Bridge, Seville, Spain, 1992

Application in buildings:

Figure 1-6: Fayetteville Festival Park, North Carolina, USA, 2004; the canopy was fabricated from tapered wide flanged steel sections

Figure 1-7: Noor-e-Mobin Sports Hall, Semnan, Iran, 2010; typical tapered steel member frame structure

Figure 1-8: 440 House, Palo Alto, California, USA, 2000; Tapered beam application in a modern resident

Application of tapered steel columns:

Figure 1-9: Forum at the Eckenberg Academy, Adelsheim, Germany 2013; tapered steel columns integral to the curtain wall façade transfer the remainder of the roof loads

1.4: Outline of this dissertation

Chapter 1: Introduction and justification of the importance of the project and objectives
Investigate the most influential parameters and understand the Eurocode background.

Chapter 2: Literature review
Review of the existing studies in the field of tapered sections is presented. The analytical background for uniform members is given and used as the benchmark or starting point for tapered member case to be developed. The General Method is adopted and applied to both uniform and non-uniform member, wherefore, the result can be analysed. This chapter also includes the assessment of journal articles, books and webpages

Chapter 3: Main description of the work done

Chapter 4: Results and discussion for lateral-torsional buckling of uniform beam
Analytical model is developed and verified. M_{cr} is analysed.

Chapter 5: Results and discussion for lateral-torsional buckling of tapered beam

Chapter 7: Conclusions

Chapter 2

2.0: LITERATURE REVIEW

2.1: Structural analysis

In the design procedure, the internal (member) forces and moments within the structure need to be determined from a global analysis before checking the strength of cross-sections and the stability of members. For a non-uniform member, computer analysis is necessary. Figure 2-0 shows a graph developed from the different methods of global analysis and elastic buckling load is also illustrated.

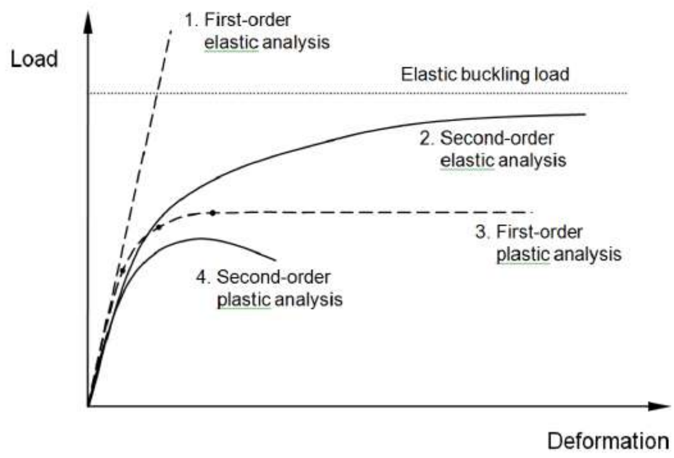


Figure 2-0: Load-deformation for type of analysis

There are four types of global analysis given in Table 2-0 (below),

		Take to account of
First order analysis	Elastic	initial geometry and fully linear material behavior
	Plastic	initial geometry and non-linear material behavior
Second order analysis	Elastic	deformed geometry and fully linear material behavior
	Plastic	deformed geometry and non-linear material behavior

In addition, there have been many studies involving the elastic behavior and plastic stability issues of tapered sections; these included experimental, analytical and numerical approaches. The types of analysis procedures shown in Figure 2-0 and Table 2-0 are described and illustrated in Section 2.3.

2.2: Tapered section

The variation of the h/b of the cross-section in a tapered section relatively to a uniform section leads to a differences in the stress determination (when analyse with Euler-Bernoulli theory for uniform section). When there is an increase in the angle of taper of the beam α, there are additional normal stresses and shear stresses. Stresses are developed perpendicularly to the inclination of the flange (as shown by the dotted in Figure 2-1). These stresses can be determined by the analytical solutions developed by Timoshenko and Goodier, 1970 [4]. The direction and equilibrium of forces in a tapered section is shown with a black arrow on Figure 2-1.

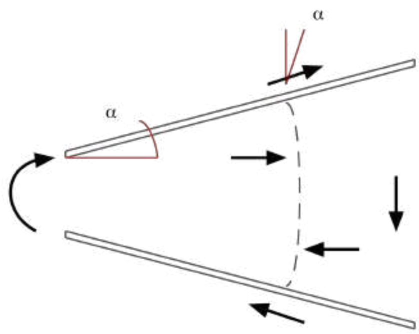


Figure 2-1: Forces in a tapered section

Furthermore, when α is relatively small (α < 15°), the stress difference between the tapered section and the uniform section is negligible, hence, the formulae for uniform member can be used to design a tapered member (Galambos, 1988).

Butler and Anderson (1963) have conducted experiments on elastic stability of web and flange tapered beams. These included an investigation of bracing requirements. Prawel et al[1]. (1974) [5] analysed members with inelastic stability, where the measured residual stresses of tapered sections showed stress distribution, similar to that of the residual stresses of uniform sections with welded cross sections.

[1] et al. is used as an abbreviation of 'et alii' meaning and others

2.3: Stability verification procedures for tapered sections

2.3.1: Stability verification procedure for a tapered column

The expressions concerning the stability of tapered columns are mostly applied to tapered beam aswell, where formulae for calculation of elastic critical forces are used. Hirt and Crisinel (2001) [6], which they determined an expression for finding the equivalent inertia of I-cross section tapered columns I_{eq} . Lee (1972) [7] presented an expression for modification factor g of the tapered member's length. The critical load's calculation is based on the smallest cross section. Petersen (1980) [8] developed design charts for the extraction of a factor β to be applied to the critical load of a column with the same length and the smallest cross section are available for different boundary conditions and cross section shapes. In addition, Baptista and Muzeau (1998) [9] provide a stability verification of tapered columns, where the coefficient K is applied to the reduction factor of a column with the smallest cross section.

AISC (2010) [10] present an equivalent uniform section, which has the same critical load and first order resistance and can be verified considering using EC3.

2.3.2: Stability verification procedure for a tapered beam

Kitipornchai and Trahair (1972) [11] present an analytical solution for the elastic critical moment M_{cr} for the tapered beams for any type of tapered I-beam loading. Galéa (1986) provides expressions for elastic critical moment, which was given by the elastic critical load of a web-tapered beam subject to a uniform bending moment distribution.
In addition, in Ibañez and Serna (2010) [12], M_{cr} is given based on the C_1 factors presented.
In this approach (known as the 'Equivalent Moment Approach', an equivalent uniform beam replaces the tapered beam by modifying the bending moment diagram. Where the tapered beam subjected to M(x) is replaced by a uniform beam with the smallest cross section, the new moment M*(x) acting at each cross section of this equivalent beam is given by considering the critical moment which would be obtained at each cross section of the tapered beam $M_{cr}(x)$, where:

$$M^*(x) = M(x) \cdot \frac{M_{cr,0}}{M_{cr}(x)}$$

$M_{cr,0}$ is the critical moment obtained by the smallest cross section. This means that tan equivalent uniform beam with a distribution of moments given by M*(x) is obtained. With an

adequate C_1 factor for this moment distribution and using the formula for uniform beams in EC3, The critical moment of the tapered beam can be determined.
Andrade et al. (2005) [13], presents an expression for the calculation of M_{cr} based on the Rayleigh-Ritz method. C_1 factor is calibrated for the case of tapered beams with fork conditions and subjects to the end moments. Moreover, an expression for M_{cr} of tapered beams subjects to concentrated load is also given in Andrade et al. (2007). Horne et al. (1979) [14] present expressions for the calculation of M_{cr} for tapered section with partial bracing near the tension flange.

Boissonnade (2002) and Andrade et al (2007) [15] refer to the inadequacy of using finite stepped uniform beam elements to analyse of tapered member stability (not taken into account of the inclination of the flange). As a result, adequate elements to account for the torsional behavior of tapered members are developed.

For plastic analysis (geometric non-linearity is taken into account), the following research is as follows. In AISC (2010), the mapping of the elastic buckling strength of tapered members to the design strength of an equivalent uniform section is performed (uniform beam with first order resistance and M_{cr} is determined and then EC3 is applied to the beam). Bradford (1988) provides a finite element for the elastic buckling resistance of the tapered double symmetric I-beams, under the action of end moments or uniformly distributed load. In Vandermeulen (2007), solutions for a ''plateau'' slenderness λ_0 or the limit slenderness (for which instability effects will influence the resistance of the beam) is given. The adequate imperfection factors α are also given for analysed cases with linear bending moment distributions. With λ_0 and α being calibrated for a range of tapering and loading situations, EC3 can be applied to provide adequate design.

More importantly, a suitable moment of inertia for tapered members is provided by Hirt and Crisinel (2001). Here, the moment of inertia for tapered member, $I_{eq,}$ depending on the type of web variation is given by:

$$I_{y,eq} = CI_{y,max} \quad (2.0)$$

Where,

$$C = 0.08 + 0.92r$$

$$r = \sqrt{I_{y,min}/I_{y,max}}$$

2.3.3: Stability verification procedure for a tapered beam-column

The stability verification is performed on the basis of the interaction formula for uniform section with the provisions for the tapered beams and columns. For beam-column, General Method can be used with the generalized slenderness applied in the Ayrton-Perry equation. Thus, the most restrictive buckling curve for lateral-torsional buckling in Clause 6.3.1 or Clause 6.3.2 is adopted. Moreover, in the process of performing cross-sectional checks, a second order elastic analysis (see Table 2-0), where local second order effects and imperfections are considered. This is because there are no satisfactory section stability verification procedures for non-uniform members, thus, providing an over safe results. Using the General Method and consideration of certain buckling curves which are assumed to be adequate can even lead to an unsafe results due to the complexity of the imperfections. Therefore, all second order effects need to be accounted for in the structural analysis such that only cross section checks need to be performed.

2.4: Application of EC3 methodologies for uniform sections

2.4.1: Uniform members in bending (clause 6.3.2)

Flexural member such as beams are the most common type of structural member. Essentially, beams are intended to span across two supports and transmit the loads mainly by bending action.

Lateral-torsional buckling verification of beams is performed according to clause 6.3.2. Buckling resistance is determined by using the buckling curve (Figure 2-2) for flexural buckling. The non-dimensional slenderness for beams is denoted by $\bar{\lambda}_{LT}$, this is to characterize lateral-torsional buckling. On one hand, beams with low value of $\bar{\lambda}_{LT}$ will failure by material yielding or in-plane failure[2]. On the other hand, beams with high value of $\bar{\lambda}_{LT}$ will failure by lateral-torsional buckling.

As well as lateral-torsional buckling, there are other common checks that should be carried out to verify the suitability of a beam to support the applied loading; these include bending moment resistance, shear resistance and deflections.

2 excessive bending and deformation in the plane of the applied loading

For laterally unrestrained beams, lateral-torsional buckling *must* be checked and designed for according to EC3. EC3 provides design methods that cover the areas that can influence lateral-torsional buckling. These include section shape, beam geometry, the degree of lateral restraint, support conditions, type of loading, the residual stress pattern and initial imperfections. As well as checking in-plane bending, EC3 presents two approaches for the design check of lateral-torsional buckling effect of I-beams.

The first check given in Clause 6.3.2 involves buckling resistance, where the in-plane bending of beams is checked.

The design buckling resistance of a laterally unrestrained beam is given by EC3:

$$M_{b,Rd} = \chi_{LT} \frac{W_y f_y}{\gamma_{m0}} \tag{2.1}$$

Where,

$$M_{y,Rd} = \frac{W_y f_y}{\gamma_{m0}} \tag{2.2}$$

Thus, substituting equation (2.2) to $M_{b,Rd}$ gives:

$$\mathbf{M_{b,Rd} = \chi_{LT} M_{y,Rd}} \tag{2.3}$$

The design buckling resistance $M_{b,Rd}$ moment is equal to reduction factor for lateral-torsional buckling χ_{LT} times design bending moment resistance in y-y axis $M_{y,Rd}$. $M_{y,Rd}$ can be determined by considering an adequate section properties according to the respective cross-section class. In equation (2.1):

- $W_y = W_{pl}$ for Class 1 or 2 cross-sections
- $W_y = W_{el}$ for Class 3 cross-sections
- $W_y = W_{eff}$ for Class 4 cross-sections

Moreover, when designing to EC3 for lateral-torsional buckling, lateral-torsional buckling curves for uniform members are identify by equation (2.4). This can be applied to all section types.

The reduction factor for lateral-torsional buckling can also be given by:

$$\chi_{LT} = \frac{1}{\Phi_{LT} + \sqrt{\Phi_{LT}^2 - \bar{\lambda}_{LT}^2}} \quad (2.4)$$

Where,

$$\Phi_{LT} = 0.5[1+\alpha_{LT}(\bar{\lambda}_{LT} - 0.2) + \bar{\lambda}_{LT}^2] \quad (2.5)$$

In equation (2.4), α is an imperfection factor calibrated both by extensive numerical and experimental parametric tests (Beer and Schulz, 1970 [16]). This was adopted in EC3 in the Ayrton-Perry format. The values for α and the corresponding buckling curves are shown in Figure 2-2 and Table 2-1. This is also known as the reduction curve.

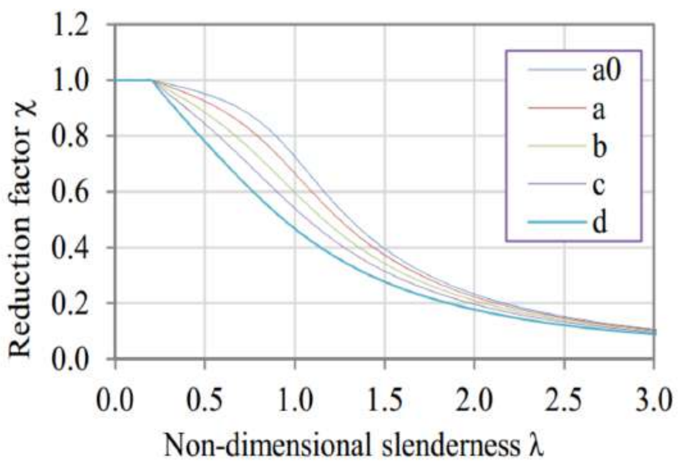


Figure 2-2: Buckling curve (L.S. Marques (2012))

Table 2-1: Imperfection factors α for buckling curves (Table 6.1 of EC3-1-1)

Buckling curve	a_0	a	b	c	d
Imperfection factor α	0.13	0.21	0.34	0.49	0.76

Table 2-2: Selection of buckling curve for different cross-section types (Table 6.5 of EC3-1-1)

Fabrication procedure	h/b limits	Buckling about axis	Buckling curve
Rolled I-sections	>1.2	y-y	a
		z-z	b
	≤1.2	y-y	c
		z-z	c
Welded I-sections	-	y-y	b
		z-z	c

and,

$$\bar{\lambda}_{LT} = \sqrt{\frac{W_y f_y}{\mathbf{M_{cr}}}} \tag{2.6}$$

Once the buckling curve is picked, Table 2.1 is used to determine the value of imperfection factor α_{LT} (Table 2-1), thus, reduction factor χ can be calculated. The member lateral-torsional buckling resistance $M_{b,rd}$ (equation (2.1)) can be checked (must be larger or equal to M_{ed}). If $M_{b,rd}$ is less than M_{ed}, this means that the designed section is inadequate, therefore, another cross-section must be chosen for design.

2.4.2: Elastic critical moment for lateral-torsional buckling M_{cr}

As shown on the equation (2.6), the determination of non-dimensional lateral-torsional buckling slenderness $\bar{\lambda}_{LT}$ first requires calculation of elastic critical moment for lateral-torsional buckling M_{cr}. However, EC3 does not provide any guidance on how M_{cr} should be calculated, hence mentioning that it should be based on the gross cross sectional properties and take into account the loading conditions, the real moment distribution and the lateral restraints. Reasons for the omission of such formulations include the complexity of the subject and lack of consensus between the contributing nations; by many, it is regarded as 'textbook material'.

In this dissertation, the expression for M_{cr} is provided by the NCCI[3] and ENV 1993-1-1 (1992) for lateral-torsional buckling (equation (2.7)).

[3] Non-Contradictory Complementary Information, Access Steel SN003a-EN-EU 2007, equation 3

$$M_{cr,0} = \frac{\pi^2 EI_z}{L_{cr}^2}\left[\frac{I_w}{I_z}+\frac{L_{cr}^2 GI_T}{\pi^2 EI_z}\right]^{0.5} \quad (2.7)$$

For uniform doubly-symmetric cross-sections which is loaded through the shear centre at the level of the centroidal axis. M_{cr} is given in the expression below (equation (2.8))

$$M_{cr} = C_1 \frac{\pi^2 EI_z}{L_{cr}^2}\left[\frac{I_w}{I_z}+\frac{L_{cr}^2 GI_T}{\pi^2 EI_z}\right]^{0.5} \quad (2.8)$$

Where, warping constant I_w in the above cases is:

$$I_w = \frac{h_w^2 I_z}{4}$$

Torsional constant I_T is given by:

$$I_T = \frac{2b_f t_f^3 + h_w t_w^3}{3}$$

Correction factor C_1 is determined from Table 2-3 for end moment loading and Table 2-4 for transverse loading. C_1 factor takes into account the effect of the bending moment diagram. C_1 factor can also be used to modify $M_{cr,0}$, for example $M_{cr} = C_1 M_{cr,0}$ when a uniform moment is applied to the beam..

Furthermore, for end moment loading, the value of C_1 may be approximated by equation (2.9)

$$.C_1 = 1.88 - 1.40\psi + 0.5\psi^2 \qquad C_1 \leq 2.70 \quad (2.9)$$

Table 2-3: C_1 values for end moment loading

Loading and support conditions	Bending moment diagram	Value of C_1
M ψM	$\psi = +1$	1.000
	$\psi = +0.75$	1.141
	$\psi = +0.5$	1.323
	$\psi = +0.25$	1.563
	$\psi = 0$	1.879
	$\psi = -0.25$	2.281
	$\psi = -0.5$	2.704
	$\psi = -0.75$	2.927
	$\psi = -1$	2.752

Table 2-4: C_1 values for transverse loading

Loading and support conditions	Bending moment diagram	C_1
w		1.132
w		1.285
F		1.365
F		1.565
F ℄ F; = = = =		1.046

In conclusion, for uniform member, once M_{cr} is found by using equation (2.7), non-dimension slenderness $\bar{\lambda}_{LT}$ can be calculated using equation (2.6). Thus, the buckling curve can be selected and the imperfection factor χ_{LT} is found. In results, $M_{b,Rd}$ is calculated by equation (2.1). This is then checked if greater than M_{ed}.

2.5: Application of EC3 methodologies for non-uniform sections (tapered member)

For non-uniform member, once M_{cr} is found by FEA[4], non-dimension slenderness $\bar{\lambda}_{LT}$ can be calculated by using equation (2.14). Therefore, by applying the General Method to non-uniform beam, the selection for lateral-torsional buckling curve should be based on Table 2-5 (Table 6.5 of EC3-1-1), where buckling curve c or d is chosen for welded I-section. Buckling curve c and d has an imperfection factor α_{LT} of 0.49 and 0.76.

Table 2-5: Selection of buckling curves for lateral-torsional buckling

Fabrication procedure	h/b	General Case	Special Case
Rolled I-sections	≤2	a	b
	>2	b	c
Welded I-sections	≤2	c	c
	>2	d	d
Other cross-sections	-	d	d

In addition, Clause 6.3.2.3 provides an expression for non-uniform members. Which, lateral torsional buckling curves can be described as:

$$\chi_{LT} = \frac{1}{\Phi_{LT} + \sqrt{\Phi_{LT}^2 - \beta\,\bar{\lambda}_{LT}^2}} \qquad (2.10)$$

4 Finite Element Analysis

where,

$$\Phi_{LT} = 0.5[1+\alpha_{LT}(\bar{\lambda}_{LT} - \bar{\lambda}_{LT,0}) + \beta\,\bar{\lambda}_{LT}^{2}] \qquad (2.11)$$

$\beta = 0.75$ (Recommended value)

$\bar{\lambda}_{LT,0} = 0.4$ and $\bar{\lambda}_{LT}$ is calculated with the obtained M_{cr} by equation (2.14)

This means that, for a given non-uniform section, $\bar{\lambda}_{LT}$ can be calculated. This is then used to obtain χ_{LT}.

Once χ_{LT} is found $M_{b,Rd}$ for non-uniform member can be calculated for designing purposes.

2.5.1: Elastic critical moment for lateral-torsional buckling M_{cr} for tapered member

Andrade et al. (2005) provided the following expression for the determination of the critical moment of web-tapered beams subject to a linear bending moment distribution:

$$M_{cr} = C_1 \left[\frac{\pi}{L} + \sqrt{1+ak^2_{w,h\,max}+bak^2_{it,h\,max}}\;\sqrt{EI_z I_{T,h\,max}}\right]^{0.5} \qquad (2.12)$$

where,

$\gamma_h = h_{max}/h_{min}$

$$a = 1-1.021*\left(1-\frac{1}{\gamma_h}\right)+0.2927*\left(1-\frac{1}{\gamma_h}\right)^2$$

$$b = -0.3815*\left(1-\frac{1}{\gamma_h}\right)$$

$$I_{T,h\,max} = \frac{2b}{3}\left(\frac{t_f}{\cos\alpha}\right)^3 + \frac{h_{max}t_w^3}{3}$$

$$I_z = \frac{1}{6}b^3 t_f \cos^3 \boldsymbol{\alpha}$$

$k_{w,h\,max}=\frac{\pi}{L}\sqrt{\frac{EI_{w,h\,max}}{GI_{T,h\,max}}}$, with $I_{w,\,h\,max}=\frac{1}{24}h_{max}^2\,bt_f\cos^3\alpha$

$k_{It,h\,max}=\frac{h_{max}t_w^3}{3I_{T,\,h\,max}}$

α is the taper angle

C_1 is given by:

$$C_1=\frac{1}{\sqrt{c+d(1-\psi)+f(1-\psi)^2}}$$

which,

$\psi = M_{y,Ed,hmin}/M_{y,Ed,hmax}$

For $-0.5< \psi <1$:

$c = 1$

$$d= -1.2060+ \frac{0.2160}{\gamma_h}+0.2275e^{\frac{-3}{2k_{w,\,h\,max}}}-0.2090\times\frac{e^{\frac{-3}{2k_{w,h\,max}}}}{\gamma_h}$$

$$f= 0.3973- \frac{0.1174}{\gamma_h} - 0.100e^{\frac{-3}{2k_{w,\,h\,max}}} - 0.1070\times\frac{e^{\frac{-3}{2k_{w,h\,max}}}}{\gamma_h}$$

For $-1< \psi <1/2$:

$$c= -1.4340+ \frac{0.3748}{\gamma_h}-0.1828e^{\frac{-3}{2k_{w,\,h\,max}}}-0.2770\times\frac{e^{\frac{-3}{2k_{w,h\,max}}}}{\gamma_h}$$

$$d= -1.6930+ \frac{0.6487}{\gamma_h}+0.5275e^{\frac{-3}{2k_{w,\,h\,max}}}-0.5655\times\frac{e^{\frac{-3}{2k_{w,h\,max}}}}{\gamma_h}$$

$$f= 0.5628- \frac{0.2373}{\gamma_h}-0.2208e^{\frac{-3}{2k_{w,\,h\,max}}}-0.2220\times\frac{e^{\frac{-3}{2k_{w,h\,max}}}}{\gamma_h}$$

Chapter 3

3.0: Main description of the work done

General Method, Special case (Clause 6.3.2.3) was applied to a modeled tapered section. A value of M_{cr} was obtained and a reduction curve is plotted. This was then compared with the reduction curve given in the Eurocode 3.

3.1: Linear analysis of beam by FEM

Commercial finite element package ABAQUS v6.13 was used to provide an analysis of the strength capacity of beams. By using equation (2.8), a value of M_{cr} for any individual beam can be found and compared with the value of M_{cr} given by the finite element analysis.

When analysing the beam, the following aspects have been taken into account:

- modeling of the structure or structural component, including boundary conditions and type of element
- modeling of materials' properties
- unit used throughout the analysis (must be constant)
- modeling of loads
- choice of software

2 types of element were used in modeling the uniform beam, these include shell elements and beam (wire) elements. Shell element is used for modeling tapered section.

Linear analysis is performed, in analysis step (Figure 3-1); buckling analysis in linear perturbation procedure type is chosen. By using this type of analysis, the linear critical moment can be found. This is calculated by multiplying the value of the applied reference moment M_{ref} by the obtained eigenvalue λ:

$$\mathbf{M_{cr} = \lambda M_{ref}} \qquad (3.0)$$

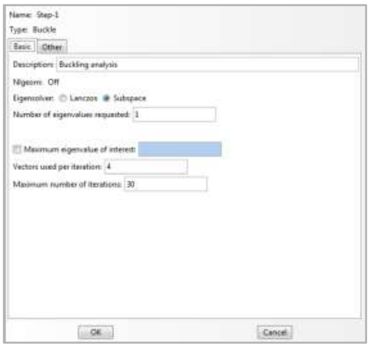


Figure 3-1: Step used for linear buckling analysis

According to the Abaqus manual, eigenvalue buckling analysis:

- is generally used to estimate the critical (bifurcation) load of "stiff" structures;
- is a linear perturbation procedure;

When moment is applied instead of load, critical moment M_{cr} is given.

3.2: Modeling

The shell element model was modeled in a 3D modeling space using a deformable planar type. To ensure that the model characterises the beams real behavior, the boundary conditions, loading, materials used must be inputted correctly in the model. The unit was kept constant throughout the analysis (metre).

Moreover, a reference point was created at the centroid of the beam (Figure 3-2). A constraint was then applied to the reference point, causing the reference point to acts as a node. Boundary conditions and loads can be applied to the reference point. This was to make sure that the load is acted upon the whole cross-section.

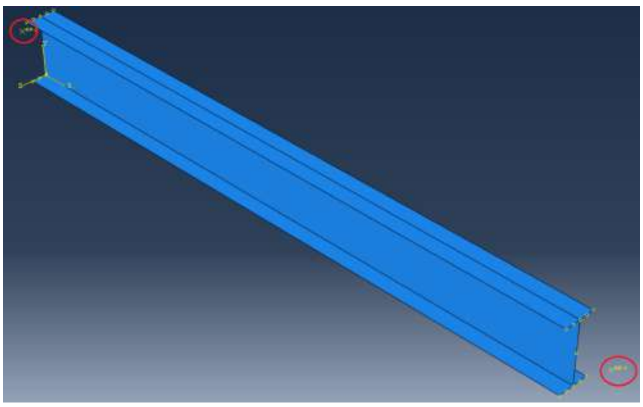


Figure 3-2: Reference points (marked in red)

3.2.1: Material properties

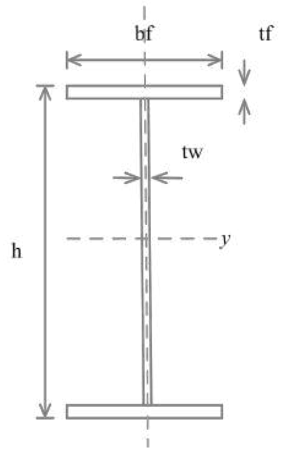
S355 steel grade was used, with yield stress of 355MPa. A modulus of elasticity E of 210GPa and a Poisson's ratio of 0.3 were adopted.

3.2.1: Support conditions

The boundary conditions for a simply supported single span member with end fork conditions are implemented in the wire planar model (Section 3.3). Fixed support with free end was applied for shell models. For the shell element, the following restraints were imposed on the free end: vertical, transverse displacement and rotation about x-x axis at the free node. Cross-sections at the node at the free-end are modeled to remain straight, however, allowing for warping. This means that the flanges can move independently from the web.

3.3: Uniform beam 1

3D models consist firstly of wire planar (beam) type element and followed by shell elements were used for uniform beam in bending. Figure 3-3 illustrates the cross-section of the beam HEA300. Four HEA300 beams with length equals to 2, 4, 8 and 12 metres have been modeled for the beam element. Where, beam element type BS31OS have been adopted. According to the ABAQUS User's Manual, this is the 'Timoshenko' type beam. All models were analysed by meshing with the deviation factor equal to 0.1.



b_f = 300mm
t_f = 14mm
t_w = 8.5mm
h = 290mm

E = 210GPa
G = 81GPa
f_y = 355MPa

Figure 3-3: HEA300 cross-section

The material properties and cross-section is given in Figure 3-3. The three-points (default) Simpson thickness integration rule has been adopted for each segment, making up the section. Beams are simply supported and subjected to uniform moment at both ends of the beam (Figure 3-4).

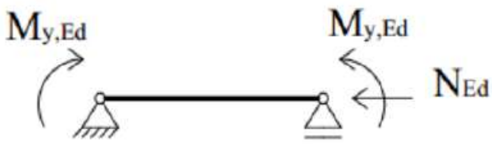


Figure 3-4: Pin-ended beam (L.S. Marques (2012),

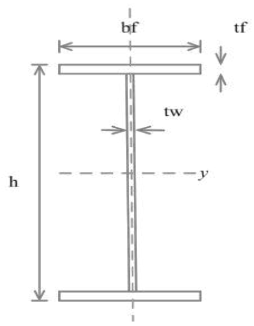
M_{cr} can be achieved by multiplying the value of applied reference load by obtained eigenvalue (Sabat 2009 [17]), as shown in equation (3.0).

When the eigenvalue is obtained, M_{cr} for wire planar element can be calculated, this is compared with the theoretical critical moment calculated by equation (2.8).

A shell element with HEA300 beam profile is created for further analysis of the critical moment. *This is to ensure that the method of modeling shell element will give the correct value of M_{cr} when modeling with tapered section.*

3.4: Uniform beam 2

Lastly, a uniform I-beam with cross-section h=200mm and b=100m (Figure 3-5) was modeled and analysed. The models have a fixed support at one end with the following restraint at the other end: vertical, transverse displacement and rotation about xx axis. This beam has a similar h_l value to the web tapered-beam. This is to be compared with the M_{cr} value given by the tapered beam.



b_f = 100mm
t_f = 10mm
t_w = 6mm
h = 200mm

E = 210GPa
G = 81GPa
f_y = 355MPa

Figure 3-5: Analysed I beam

3.5: Web-tapered beam

Shell element was used to create a typical tapered section. Tapered I-beam with a taper ratio γ_h = 2 is modeled. The cross-section at the maximum height h_{max} is similar to Figure 3-5 and at the minimum height h_{min}, h_{min} = 100mm. Figure 3-6 presents a diagram of the tapered beam which was modeled.

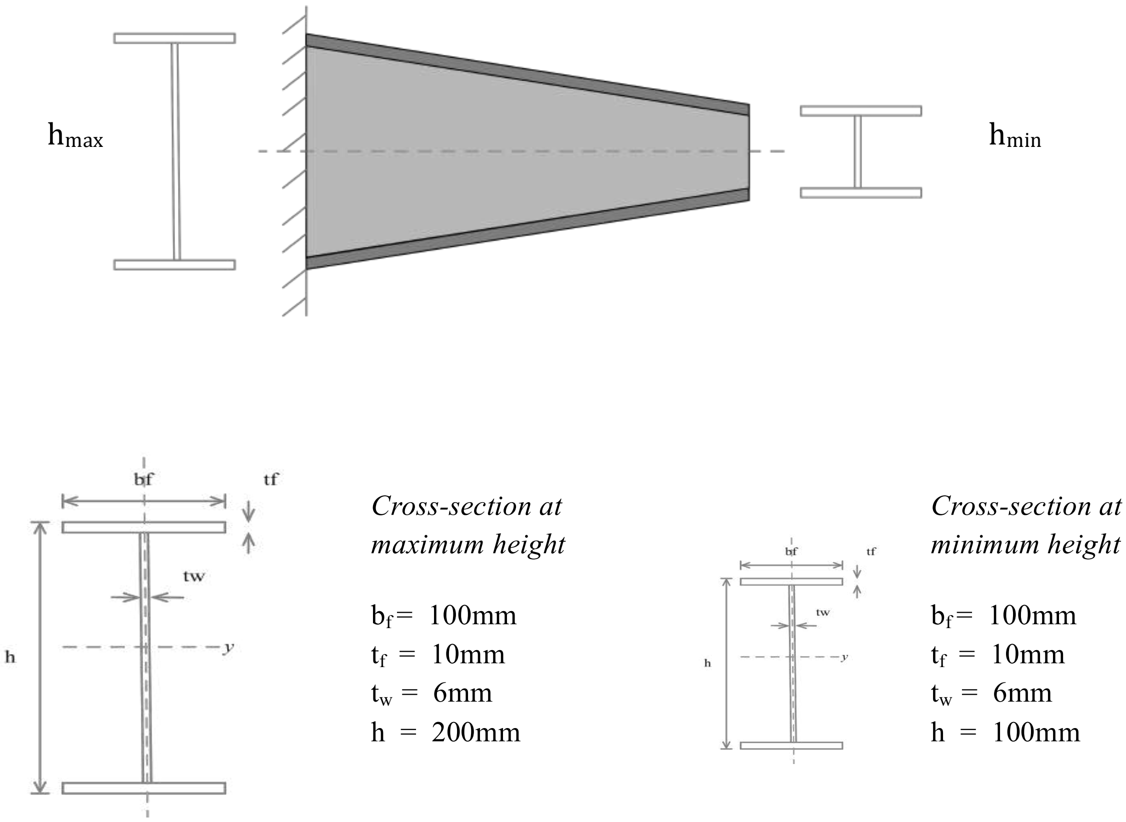


Figure 3-6: Tapered I-beam γ_h = 2

The models was analysed by FEM with the approximate global mesh size of 0.025, 0.01, 0.0075 and 0.0025. In mesh controls, quad element shape with structured technique was selected (Figure 3-7).

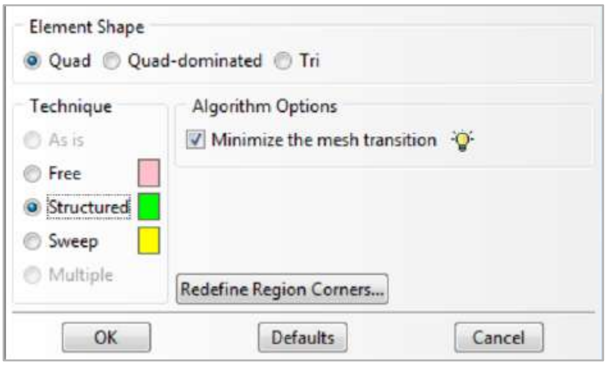


Figure 3-7: Mesh Controls

The average eigenvalue from the analysis was obtained. M_{cr} can be achieved by multiplying the value of applied reference load by the obtained eigenvalue as shown on equation (3.0) (Sabat 2009).

Once M_{cr}[5] is calculated, equation (2.0) was used to find $I_{y,eq}$. General Method Clause 6.3.2.3 (Special Case) is tested. This was achieved by using equation (2.6) to find $\bar{\lambda}_{LT}$ and this can be substituted into equation (2.11), and resulting in χ_{LT} to be calculated. A graph of $\bar{\lambda}_{LT}$ vs χ_{LT} was plotted. This is known as the reduction curve.

The value of M_{cr} was also compared with the theoretical value of M_{cr} for the Uniform beam 2.

Buckling curve of γ_h= 2 (γ_h= h_{max} /h_{min}) is plotted as a result, using Clause 6.3.2.3 (equation (2.10) and (2.11). This is then put in contrast with other buckling curves provided by the General Method (Special Case) in the Eurocode 3.

[5] W_y is assumed to be the same as the uniform section

Chapter 4

4.1: Results - Uniform beam

4.1.1: Wire type element model HEA300

Table 4.1: Theoretical critical buckling moment: *ENV 1993-1-1 (1992) (equation 2.7) vs Finite Element Analysis*

Beam HEA300 length	Elastic Critical Moment M_{cr} (kNm)	
	ENV 1993-1-1 (1992)	FEA
2m	4458.0	4610.4
4m	1241.0	1287.4
8m	413.2	422.9
12m	241.3	244.9

**Calculation in Appendix A-i)*

4.1.2: Shell type element HEA300 (Figure 3-3) – Uniform beam 1

Table 4.2: Theoretical critical buckling moment: *ENV 1993-1-1 (1992) (equation 2.7) vs Finite Element Analysis* ***(Simply supported boundary conditions)***

Beam HEA300 length	Elastic Critical Moment M_{cr} (kNm)	
	ENV 1993-1-1 (1992)	FEA
2m	4458.0	4604.4

**Calculation in Appendix A*

4.1.3: Shell type element (Figure 3-5) – Uniform beam 2

Table 4.3: Theoretical critical buckling moment: *ENV 1993-1-1 (1992) (equation 2.7) vs Finite Element Analysis* ***(Cantilever)***

Beam with cross-section shown in Figure 3-4 length	Elastic Critical Moment M_{cr} (kNm)	
	ENV 1993-1-1 (1992)	FEA
2m	209.2	667.7
6m	50.04	114.5
12m	23.9	46.4

**Calculation in Appendix A-ii)*

4.2: Discussion - Uniform beam

Uniform beam 1:

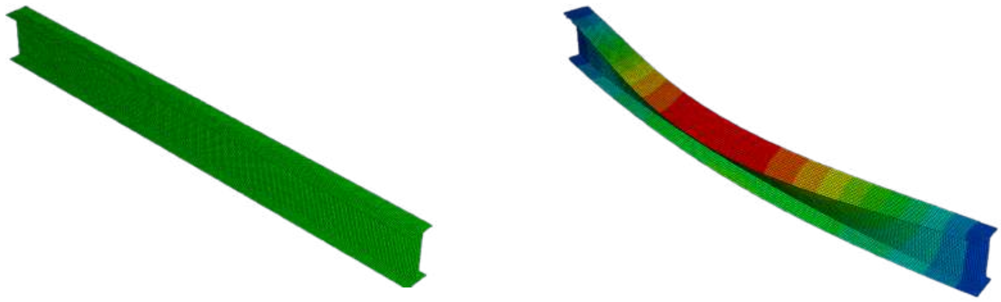
Table 4.1 shows that Finite Element Analysis provides an accurate result when compared to the hand calculations. FEA illustrates an error of 3.3%. This error occurs because when calculating the results for ENV 1993-1-1 (1992), the beam is assumed to have no residual stresses, initial imperfections and no material-nonlinearities. Moreover, structures in reality would not reach this magnitude of load due to its geometrical imperfection, residual stress and material properties.

Uniform beam 2

For Table 4.3, the FEA provides M_{cr} values which are up to three times higher than the ENV 1993-1-1 (1992). Similar to Uniform beam 1, when calculating the results for ENV 1993-1-1, the beam is assumed to have no initial imperfections.

Another FEA was made and evaluated, where beams are able to move freely (simply supported), with moment applied at both ends of the beam and a concentrated force at one end

(Figure 3-4). The value given regarding to this analysis was 185.3 kNm for 2m beam and 40.7 kNm for 6m beam.



(a) (b)

Figure 4-1: Model and deformation of Uniform beam 2

Figure 4-1 provides a finite element model before (Figure 4-1 (a)) and after the analysis (Figure 4-1 (b)). The FEM model shown above does not provide a behavior of lateral-torsional buckling, however, when the beam in a cantilever situation, lateral-torsional buckling can be observed. Thus, the results for simply-support condition were not considered.

Chapter 5

5.1: Results – Tapered beam

5.1.1: Tapered section with $\gamma_h = 2$ (Figure 3-6)

M_{cr} taken from the average of the analysis with mesh size of 0.025, 0.01, 0.0075 and 0.0025
Angle of tapered $\alpha = 1.43^{o}$
An example of a web-tapered beam which was analysed is shown in Figure 5-1

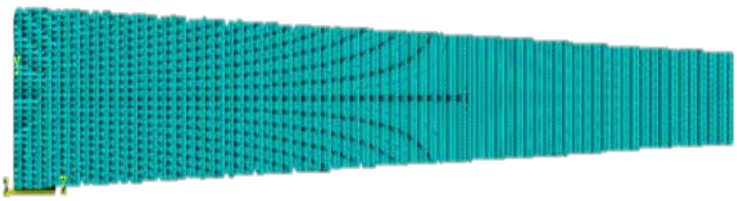
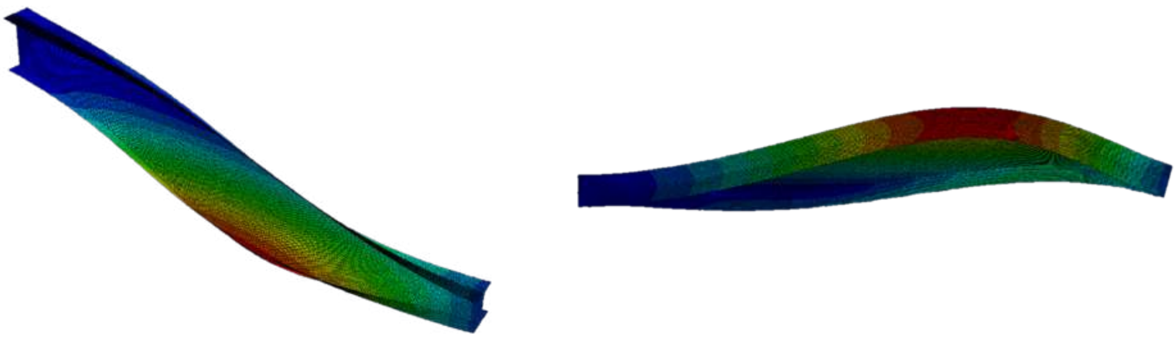


Figure 5-1: model of 2m tapered beam

Figure 5-2 (a) and (b) illustrate an example of the deformation presented by the analysis (2m length web-tapered beam).



(a) (b)

Figure 5-2: Analysed tapered beam

Figure 5-2 (a) is a 3D view of the beam with deformation. Lateral-torsional buckling can be observed in the analysis. Figure 5-2 (b) present a top view of the beam.

Table 5-1: Critical buckling moment: *Finite Element Analysis*

Tapered beam γ_h= 2	Elastic Critical Moment M_{cr} (kNm) - FEA
2m	523.0
3m	268.4
4m	173.1
6m	97.5
8m	67.5
12m	41.6
16m	11.7

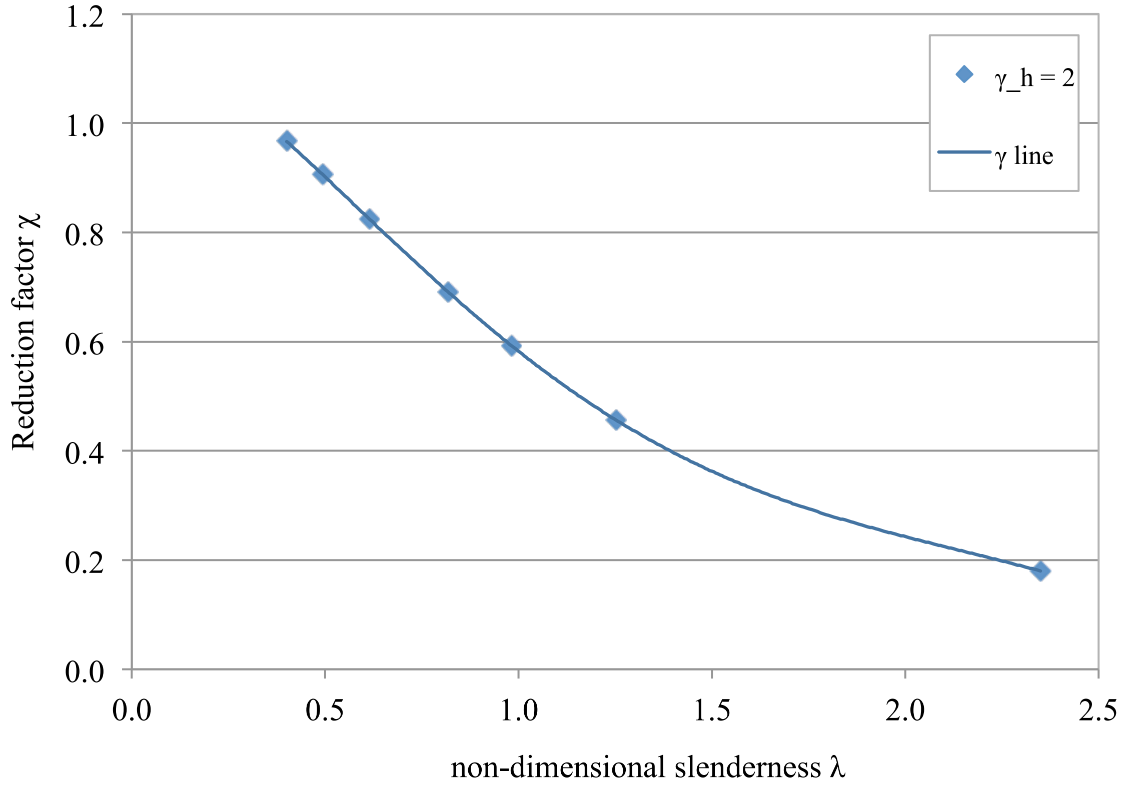
**Calculation in Appendix B – i)*

Cross-section: Class 1 (W_y is given in Appendix F of the AISC Design Guide 25)
$W_y = 237 \times 10^3$ mm^3 ,S355 grade

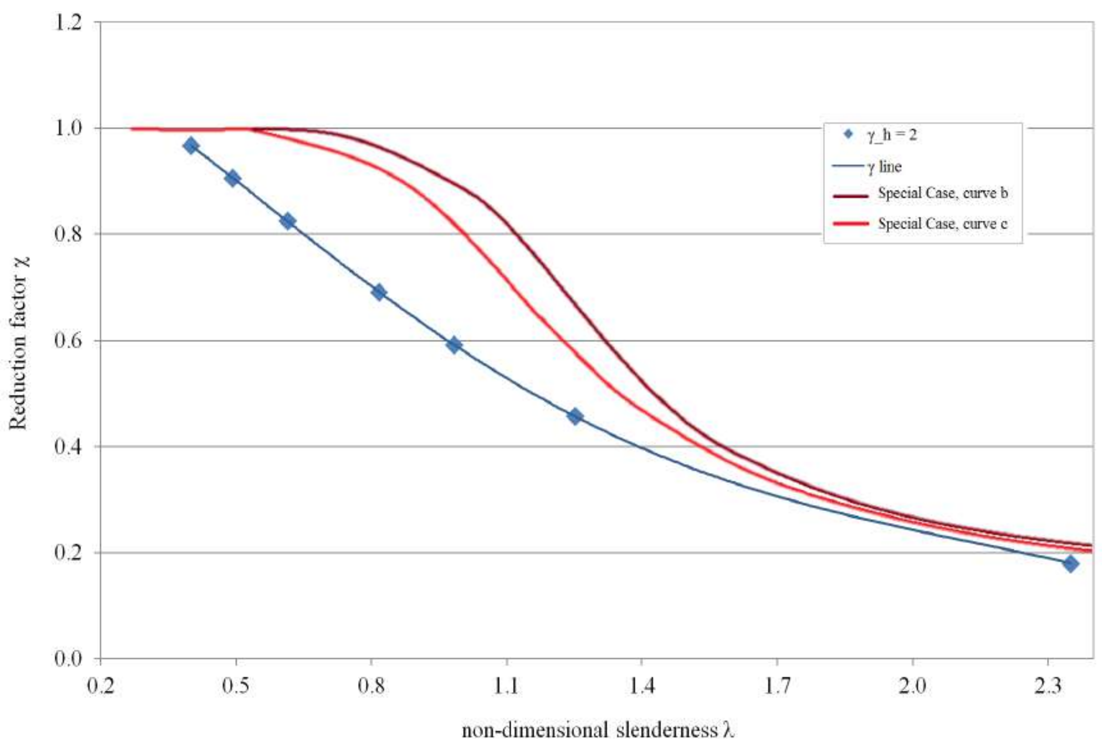
Table 5-2: M_{cr} from Table 4.3 to calculate non-dimensional slenderness $\bar{\lambda}_{LT}$, value to determine the reduction factor Φ_{LT} and reduction factor χ_{LT}: **Calculation in Appendix B – ii)*

Tapered beam γ_h= 2	$\bar{\lambda}_{LT}$	Φ_{LT}	χ_{LT}
2m	0.401	0.561	0.967
3m	0.493	0.614	0.906
4m	0.614	0.694	0.825
6m	0.818	0.853	0.692
8m	0.983	1.005	0.592
12m	1.252	1.296	0.457
16m	2.352	3.053	0.180

Graph 5-1: Reduction curve $\bar{\lambda}_{LT}$ *vs* χ_{LT} for $\gamma_h = 2$ tapered section



Graph 5-2: Reduction curve b and c for Special Case vs Graph 5-1



5.2: Discussion – Tapered beam

Table 5-1 presents M_{cr} of tapered members with taper ratio γ_h = 2. The M_{cr} for a 2m length beam has a value of 523kNm; this is 145kNm lower than the M_{cr} for uniform section. However, the percentage difference between the tapered beam and the uniform beam reduces as length increases. For a beam with 6m length, M_{cr} is 17kNm greater for the uniform section, and for a beam with 12m length, the variance is only 4.8kNm. This means that, with an increase in length, the tapered beam will have a more similar M_{cr} to the uniform section.

Furthermore, a reduction curve (Graph 5-1) is plotted using equation (2.10) and (2.11). This is compared to the original graph given in the Eurocode 3 (Graph 5-2). The reduction curve for the tapered section is lower than the reduction curve provided by Eurocode 3. The area of both curves is calculated and the percentage difference is obtained. Graph 5-2 is estimated to be 16% lower than the Special Case curve c. This means that the reduction factor χ_{LT} provided by the Eurocode 3 Clause 6.3.2.3 will gives $M_{b,Rd}$ value which is higher than the value obtained by the tapered beam analysed. Thus, this means that, when calculating the reduction factor for the tapered beam, the reduction factor given from the calculation should be reduced by 16% (equation (5.1)).

$$\chi_{LT,t} = 0.84\, \chi_{LT} \qquad (5.1)$$

In discussion, the average percentage difference between the two curves is taken for calculation. However, the graph shows a much lower value at non-dimensional slenderness $\bar{\lambda}_{LT}$ of 0.6 to 0.11. The variation of the reduction factor for Graph 5-1 can be up to 0.24 lower than the Special Case curve c. This different is significant, as when designing to the Eurocode 3 Clause 6.3.2.3, it will provides a $M_{b,Rd}$ value which will be higher than needed. In addition, plastic section modulus is assumed to be taken at the maximum cross-section area. Hence, when calculating the non-dimensional slenderness, the actual plastic section modulus of a tapered section must be used.

Moreover, to achieve a more reliable result, it is a basis to perform a second order non-linear analysis. This will take account of the second-order effects, material non-linearity, residual stresses and material's imperfections.

Chapter 6

6.0: CONCLUSIONS

- Tapered beam with taper ratio of γ_h = 2 provides a lower reduction curve when compared to the reduction curve in Eurocode 3. This is shown in Graph 5-2.
- When calculating the reduction factor for the tapered beam using Clause 6.3.2.3, the reduction factor χ_{LT}, should be multiply by a factor of 0.84 to obtain the actual reduction factor provided by the curve in Graph 5-1.
- This reduction factor will provides a lower $M_{b,Rd}$ and this will be used for designing purpose for a tapered beam.
- Second order non-linear analysis needs to be performed to take into account of the effect of residual stresses, material-nonlinearities and any imperfections.

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Figure 1-3 to 1-9 are from archdaily.com